ECE 346: Digital Signal Processing — Spring 2021
Soo Min Kwon — Week of February 8, 2021

Recitation #3: Using the Properties of DTFT

Objective & Outline

The objective of this week's recitation session is to review and use the properties of the DTFT and look at filtering of discrete-time signals (again). The following is the outline of this solution guide:

1. Problems $1-4$: recitation problems
2. Problem 5: self-assessment problem

Here are some key properties of the DTFT that we will be needing:

- **Time Reversal:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{1}$$
$$x[-n] \xleftrightarrow{\text{DTFT}} X(e^{-j\omega}) \tag{2}$$

- **Time Shifting:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{3}$$
$$x[n-n_0] \xleftrightarrow{\text{DTFT}} e^{-j\omega n_0} \cdot X(e^{j\omega}) \tag{4}$$

- **Frequency Shifting (Modulation) Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{5}$$
$$e^{j\omega_0 n} \cdot x[n] \xleftrightarrow{\text{DTFT}} X(e^{j(\omega-\omega_0)}) \tag{6}$$

- **Differentiation in Frequency Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{7}$$
$$n \cdot x[n] \xleftrightarrow{\text{DTFT}} j\frac{d}{d\omega}X(e^{j\omega}) \tag{8}$$
$$\tag{9}$$

- **Multiplication Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{10}$$
$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{j\omega}) \tag{11}$$
$$x[n] \cdot h[n] \xleftrightarrow{\text{DTFT}} \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j(\omega-\theta)}) \tag{12}$$

Note that equation (12) indicates that the multiplication of two signals in the time domain is the periodic convolution of the two signals in the frequency domain.

- **Convolution Property:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{13}$$

$$h[n] \xleftrightarrow{\text{DTFT}} H(e^{j\omega}) \tag{14}$$

$$x[n] * h[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \cdot H(e^{j\omega}) \tag{15}$$

- **Parseval's Theorem:**

$$x[n] \xleftrightarrow{\text{DTFT}} X(e^{j\omega}) \tag{16}$$

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega \tag{17}$$

The problems start on the following page.

Problem 1 (More Filtering of Signals)**.** Suppose we had the following input signal $x[n]$:

$$x[n] = \sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right) \tag{18}$$

(a) Find the DTFT of $x[n]$.

(b) Find the output of the following LTI systems with input $x[n]$:

$$h_1[n] = \frac{\sin(\pi n/6)}{\pi n} \tag{19}$$

$$h_2[n] = \frac{\sin(\pi n/6)}{\pi n} + \frac{\sin(\pi n/2)}{\pi n} \tag{20}$$

Solution:

(a) We can use Euler's formula to get the following first:

$$x[n] = \frac{1}{2j}e^{j\frac{\pi}{8}n} - \frac{1}{2j}e^{-j\frac{\pi}{8}n} - e^{j\frac{\pi}{4}n} - e^{-j\frac{\pi}{4}n}. \tag{21}$$

And then using the fact that

$$e^{j\Omega_0 n} \overset{\text{DTFT}}{\longleftrightarrow} 2\pi\delta(\Omega - \Omega_0), \tag{22}$$

The DTFT of $x[n]$ in $[-\pi, \pi]$ is:

$$X(e^{j\omega}) = \frac{\pi}{j}\delta(\omega - \pi/8) - \frac{\pi}{j}\delta(\omega + \pi/8) - 2\pi\delta(\omega - \pi/4) - 2\pi\delta(\omega + \pi/4). \tag{23}$$

(b) You should realize by now that this is simply an ideal LPF in the frequency domain:

$$H_1(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{6} \\ 0, & \frac{\pi}{6} < |\omega| \leq \pi \end{cases} \tag{24}$$

This LPF will filter out the cosine term of the input signal, as $\frac{\pi}{4} > \frac{\pi}{6}$. Thus, we're simply left with

$$Y_1(e^{j\omega}) = \frac{\pi}{j}\delta(\omega - \pi/8) - \frac{\pi}{j}\delta(\omega + \pi/8) \tag{25}$$

$$y_1[n] = \sin\left(\frac{\pi n}{8}\right). \tag{26}$$

The filter $h_2[n]$ is the sum of two ideal LPFs with cutoffs $\pi/6$ and $\pi/2$. That means in the frequency domain, we will get

$$H_2(e^{j\omega}) = \begin{cases} 2, & |\omega| \leq \frac{\pi}{6} \\ 1, & \frac{\pi}{6} < |\omega| \leq \frac{\pi}{2}. \end{cases} \tag{27}$$

Thus, the sine component gets a gain of 2 (multiply the amplitude by 2) and the cosine gets a gain of 1:

$$y_2[n] = 2\sin\left(\frac{\pi n}{8}\right) - 2\cos\left(\frac{\pi n}{4}\right) \tag{28}$$

Problem 2 (Using DTFT properties)**.** Suppose $x[n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{j\omega})$. Use DTFT properties to find the DTFTs of the following signals

(a) $x_1[n] = x[1-n] + x[-1-n]$

(b) $x_3[n] = (n-1)^2 x[n]$

Solution:

(a) We can build this up:

$$x[-n] \overset{\text{DTFT}}{\longleftrightarrow} X(e^{-j\omega}) \tag{29}$$
$$x[-n+1] = x[-(n-1)] \overset{\text{DTFT}}{\longleftrightarrow} e^{-j\omega}X(e^{-j\omega}) \tag{30}$$
$$x[-n-1] = x[-(n+1)] \overset{\text{DTFT}}{\longleftrightarrow} e^{j\omega}X(e^{-j\omega}) \tag{31}$$
$$x_1[n] \overset{\text{DTFT}}{\longleftrightarrow} 2X(e^{-j\omega})\cos(\omega). \tag{32}$$

(b) We can expand the quadratic and then use the differentiation in frequency property:

$$x_3[n] = (n^2 - 2n + 1)x[n] \tag{33}$$
$$nx[n] \overset{\text{DTFT}}{\longleftrightarrow} j\frac{d}{d\omega}X(e^{j\omega}) \tag{34}$$
$$n^2x[n] \overset{\text{DTFT}}{\longleftrightarrow} j^2\frac{d}{d\omega}\frac{d}{d\omega}X(e^{j\omega}) = -\frac{d^2}{d\omega^2}X(e^{j\omega}) \tag{35}$$
$$x_3[n] \overset{\text{DTFT}}{\longleftrightarrow} (-\frac{d^2}{d\omega^2} - 2j\frac{d}{d\omega} + 1)X(e^{j\omega}). \tag{36}$$

Problem 3 (DTFT)**.** Recall the following DTFT and its synthesis equation:

$$X(e^{j\omega}) = \sum_{n=\infty}^{\infty} x[n]e^{-j\omega n} \tag{37}$$
$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \tag{38}$$

Now consider the length-7 sequence

$$x[n] = \{1, -1, -3, 3, 7, 8, -8\}, \tag{39}$$

where the first term of the sequence (1) starts at $n = -3$. Evaluate the following without explicitly computing the DTFT of x[n].

(a) $X(e^{j0})$

(b) $X(e^{j\pi})$

(c) $\int_{5\pi}^{7\pi} X(e^{j\omega})d\omega$

Solution:

Looking at the DTFT and the synthesis equations given before this problem should help.

(a) Looking at equation (29), the DTFT formula, you should realize that $X(e^{j0})$ is simply the sum of all of the entries in $x[n]$:

$$\begin{aligned} X(e^{j0}) &= 1 + (-1) + (-3) + 3 + 7 + 8 + (-8) && (40) \\ &= 7. && (41) \end{aligned}$$

(b) Using the DTFT formula:

$$\begin{aligned} X(e^{j\pi}) &= \sum_{n=\infty}^{\infty} x[n]e^{-j\pi n} && (42) \\ &= \sum_{n=\infty}^{\infty} (-1)^n x[n]. && (43) \end{aligned}$$

Thus, $X(e^{j\pi})$ is the alternating sign based sum of $x[n]$:

$$\begin{aligned} X(e^{j\pi}) &= (-1) + (-1) + 3 + 3 + (-7) + 8 + 8 && (44) \\ &= 13. && (45) \end{aligned}$$

(c) Lastly, we can use the synthesis equation for this:

$$\begin{aligned} \int_{5\pi}^{7\pi} X(e^{j\omega})d\omega &= \int_{-\pi}^{\pi} X(e^{j\omega})d\omega && (46) \\ &= 2\pi x[0] && (47) \\ &= 6\pi && (48) \end{aligned}$$

Problem 4 (Modulation Property of the DTFT)**.** Suppose you are given the impulse response of an LTI system

$$h[n] = \frac{\sin(\pi n/4)}{\pi n}. \tag{49}$$

Plot the DTFT of $h_{\text{new}}[n]$, where

$$h_{\text{new}}[n] = e^{j\frac{\pi}{2}n} \cdot h[n]. \tag{50}$$

Solution:

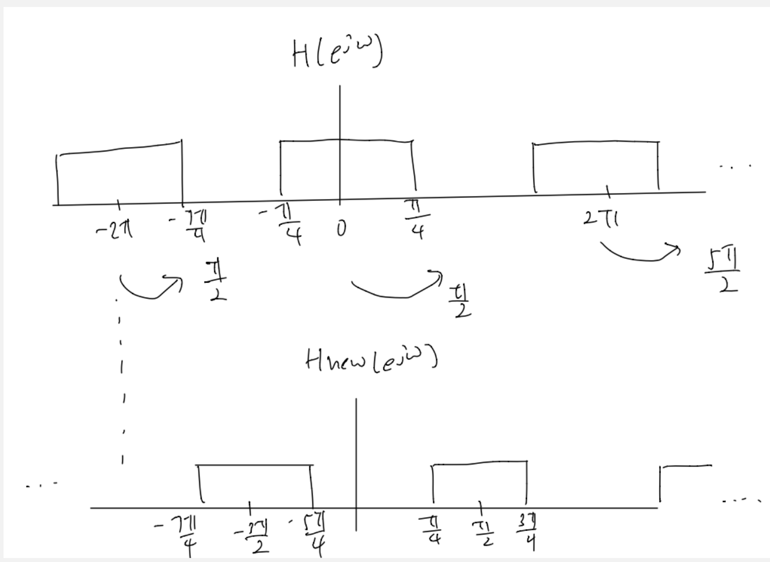
We first take the DTFT of $h[n]$:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \leq \pi \end{cases} \tag{51}$$

Now recall that the modulation property of the DTFT says that

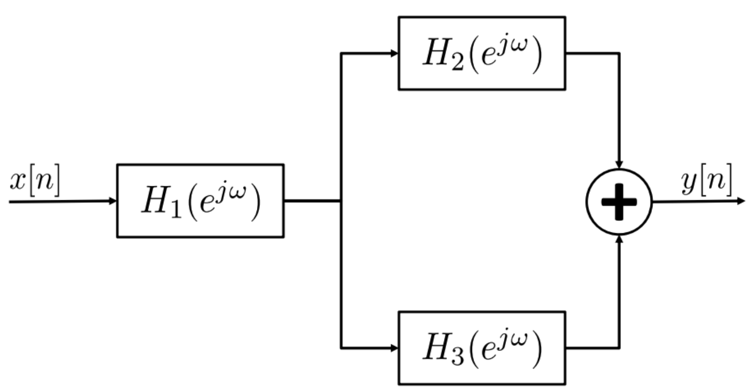
$$e^{j\omega_0 n} \cdot x[n] \xleftrightarrow{\text{DTFT}} X(e^{j(\omega-\omega_0)}). \tag{52}$$

Thus, multiplying $h[n]$ by $e^{j\frac{\pi}{2}n}$ is effectively moving $H(e^{j\omega})$ from being centered at 0 to being centered around $\frac{\pi}{2}$:



Problem 5 (Self-assessment)**.** Try to solve each problem by yourself first, and then discuss with your group.

1. **LTI Systems.** Consider the following block diagram:



Suppose $H_1(e^{j\omega})$ was an ideal **all-pass** filter with a gain of 2 (i.e. 2 from $|\omega| \leq \pi$), and

$H_2(e^{j\omega})$ was a bandpass filter with unity gain and cutoff frequencies $\frac{\pi}{6}$ and $\frac{\pi}{3}$ and $H_3(e^{j\omega})$ was a bandpass filter with unity gain and cutoff frequencies $\frac{2\pi}{3}$ and $\frac{5\pi}{6}$.

(a) Is the overall system an LTI system? Why or why not?

(b) Plot the overall system, $H_{\text{overall}}(e^{j\omega})$.

2. **Parseval's Relation.** Consider the following discrete-time signal:

$$x[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]. \tag{53}$$

Compute the integral $\int_{\pi}^{3\pi} |X(e^{j\omega})|^2$.

3. **Properties of the DTFT.** Suppose we had an LTI system with impulse response given by

$$h[n] = \frac{\sin(\pi(n-3)/4)}{(n-3)\pi}. \tag{54}$$

What is $H(e^{j\omega})$, the frequency response of this system? What type of filter is this, and what is this filter effectively doing?

Solution:

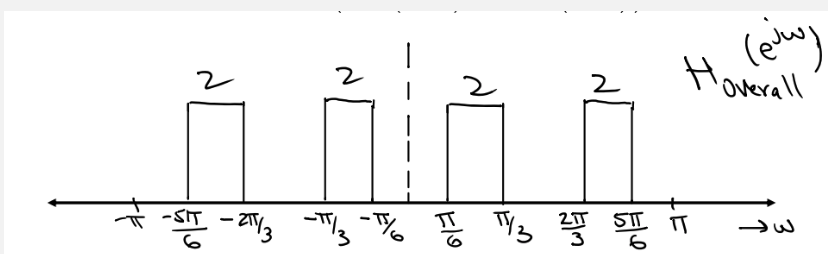
1. (a) The overall system is just a cascade and parallel combinations of LTI systems, and so the overall system, $H_{\text{overall}}(e^{j\omega})$, must also be an LTI system.

   (b) The overall system would be

$$H_{\text{overall}}(e^{j\omega}) = H_1(e^{j\omega})(H_2(e^{j\omega}) + H_3(e^{j\omega})) \tag{55}$$

$$= 2(H_2(e^{j\omega}) + H_3(e^{j\omega})). \tag{56}$$

   The plot would be



2. The main idea of this problem is to use the Parseval's Relation, that says

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2. \tag{57}$$

   The energy of this signal would be

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = 1^2 + 2^2 + 3^2 \tag{58}$$

$$= 14. \tag{59}$$

Now using the periodicity of the DTFT and Parseval's Relation:

$$\int_{\pi}^{3\pi} |X(e^{j\omega})|^2 = \int_{-\pi}^{\pi} |X(e^{j\omega})|^2 \tag{60}$$

$$= 2\pi \cdot 14 \tag{61}$$

$$= 28\pi. \tag{62}$$

3. This is an ideal low pass filter (LPF) with cutoff frequencies $[-\frac{\pi}{4}, \frac{\pi}{4}]$:

$$H(e^{j\omega}) = \begin{cases} e^{-j3\omega}, & |\omega| \leq \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\omega| \leq \pi \end{cases} \tag{63}$$

Since we are shifting in the time-domain, this is the same as adding a phase in frequency domain. Thus, the signal that exists from $[-\frac{\pi}{4}, \frac{\pi}{4}]$ will get an extra phase (if any) of $e^{-j3\omega}$.